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$\tau^* - \ddot{g}$ CLOSED SETS IN TOPOLOGICAL SPACES

T. INDIRA[1] & S. GEETHA[2]

[1]PG and Research, Department of Mathematics, Seethalakshmi Ramaswami College, Trichy, Tamil Nadu, India

[2]Department of Mathematics, Seethalakshmi Ramaswami College, Trichy, Tamil Nadu, India

ABSTRACT

In this paper a class of sets called $\tau^* - \ddot{g} -$ closed sets and $\tau^* - \ddot{g} -$ open sets and a class of maps in topological spaces is introduced and some of its properties are discussed.

KEYWORDS: cl*, τ^*-Topology, $\tau^* - \ddot{g} -$ Open Set, $\tau^* - \ddot{g} -$ Closed set, $\tau^* - \ddot{g} -$ Continuous Maps

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1. INTRODUCTION

The concept of generalized closed sets was introduced by Levine [ ]. Dunham [4] introduced the concept of closure operator cl* and a topology τ* and studied some of its properties. Pushpalatha, Easwaran and Rajarubi [11] introduced and studied τ*-generalized closed sets, and τ*-generalized open sets. Using τ*generalized closed sets, Easwaran and Pushpalatha [5] introduced and studied τ*-generalized continuous maps.

The purpose of this paper is to introduce and study the concept of a new class of sets, namely $\tau^* - \ddot{g} -$ closed sets and a new class of maps $\tau^* - \ddot{g} -$ continuous maps. Throughout this paper X and Y are topological spaces on which no separation axioms are assumed unless otherwise explicitly stated. For a subset A of a topological space X, cl (A), cl*(A) and A^C denote the closure, g-closure and complement of A respectively.

PRELIMINARIES

Definition

For the subset A of a topological space X the generalized closure operator cl* is defined by the intersection of all g-closed sets containing A.

Definition

For a topological space X, the topology τ* is defined by
$\tau^* = \{G: cl^*(G^C) = G^C\}$.

Example

Let X = {a, b, c} and τ = {x, ϕ, {a, b}}. Then the collection of subsets

$\tau^* = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ is a topology on X.

Definition

A subset A of a topological space X is called $\tau^* - \ddot{g} -$ closed set if $cl^*(A) \subseteq G$, whenever $A \subseteq G$ and G is τ^*-sg-open.

The complement of $\tau^* - \ddot{g} -$ closed set is called the $\tau^* - \ddot{g} -$ open set.

Example

Let X = {a, b, c} with $\tau = \{X, \phi, \{a\}\}$. Then the sets {X}, {$\phi$}, {b}, {c}, {a, b}, {b, c}, {a, c} are $\tau^* - \ddot{g} -$ closed sets in X.

Theorem

Every closed set in X is $\tau^* - \ddot{g} -$ closed.

Proof

Let A be a closed set.

Let $A \subseteq G$. Since A is closed, $cl(A) = A \subseteq G$, where G is τ^*-sg-open. But $cl^*(A) \subseteq cl(A)$. Thus we have $cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is τ^*-sg-open.

Hence A is $\tau^* - \ddot{g} -$ closed.

Theorem

Every τ^*-closed set in X is $\tau^* - \ddot{g} -$ closed.

Proof

Let A be a τ^*-closed set. Let $A \subseteq G$, where G is τ^*-sg-open.

Since A is τ^*-closed, $cl^*(A) = A \subseteq G$.

Thus, we have $cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is τ^*-sg-open.

Hence A is $\tau^* - \ddot{g} -$ closed.

Theorem

Every g-closed set in X is a $\tau^* - \ddot{g} -$ closed but not conversely.

Proof

Let A be a g-closed set.

Assume that $A \subseteq G$, G is τ^*-sg-open in X.

Since A is g-closed, $cl(A) \subseteq G$.

But $cl^*(A) \subseteq cl(A)$.

$\Rightarrow cl^*(A) \subseteq G$, whenever $A \subseteq G$ and G is τ^*-sg-open.

Therefore A is $\tau^* - \ddot{g} -$ closed

The converse of the above theorem need not be true as seen from the following example.

Example

Consider the topological space X = {a, b, c} with topology τ = {X, ϕ, {a}}. Then the set {a} is τ^*-$\ddot{g}$-closed but not g-closed.

Remark

The following example shows that τ^*-$\ddot{g}$-closed sets are independent from sp-closed set, sg-closed set, α-closed set, pre closed set, gs-closed set, gsp-closed set, αg-closed set and gα-closed set.

Example

Let X = {a, b, c} and {a, b, c, d} be the topological spaces.

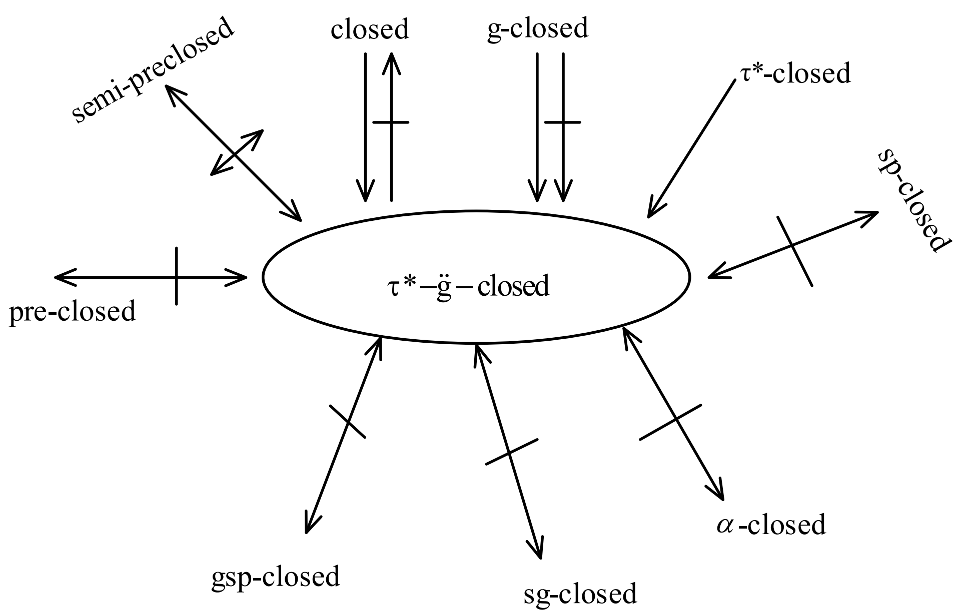
- Consider topology τ = {X, ϕ, {a}}. Then the sets {a}, {a, b} and {a, c} are τ^*-$\ddot{g}$-closed but not sp-closed.
- Consider the topology τ = {X, ϕ, {a, b}}. Then the sets {a} and {b} are sp-closed but not τ^*-g-closed.
- Consider the topology τ = {X, ϕ}. Then the sets {a}, {b}, {c}, {a, b}, {b, c} and {a, c} are τ^*-$\ddot{g}$-closed but not sg-closed.
- Consider the topology τ = {X, ϕ, {a}, {b}, {a,b}}. Then the sets {a} and {b} are sg-closed but not τ^*-$\ddot{g}$-closed.
- Consider the topology τ = {X, ϕ, {a}}. Then the sets {a}, {b}, {c}, {a,b} and {a, c} are τ^*-$\ddot{g}$-closed but α-closed.
- Consider the topology τ = {X,ϕ, {a}, {a, b}}. Then the set {b} is α-closed but not τ^*-$\ddot{g}$-closed.
- Consider the topology τ = {X,ϕ, {a}}. Then the sets {a}, {a, b} and {a, c} are τ^*-$\ddot{g}$-closed but not pre-closed.
- Consider the topology τ = {X,ϕ, {b}, {a, b}. Then the set {a} is

pre-closed but not τ*- g̈ -closed.

- Consider the topology τ = {X, ϕ}. Then the sets {a}, {b}, {c}, {a, b}, {b, c} and {a, c} are τ*- g̈ -closed but not gs-closed.
- Consider the topology τ = {X, ϕ, {b}}. Then the sets {b}, {a, b} and {b, c} are τ*- g̈ -closed but not semi pre-closed.
- Consider the topology τ = {X, ϕ, {b}, {c}, {b, c}}. The sets {b} and {c} are semi pre-closed but not τ*- g̈ -closed.
- Consider the topology τ = {X, ϕ, {a}, {b}, {a, b}}. Then the sets {b} and {a, b} are gsp-closed but not τ*- g̈ -closed.
- Consider the topology τ = {Y, ϕ, {a}}. Then the set {a} is τ*- g̈ -closed but not gsp-closed.

Remark

From the above discussion, we obtain the following implications.



A →B means A implies B, A ↛ B means A does not imply B and A ↮ B means A and B are independent.

Theorem

For any two sets A and B, cl*(A∪B) = cl*(A) ∪ cl*(B).

Proof

Since A⊆A∪B, we have cl*(A)⊂cl*(A∪B) and since B⊆A∪B, we have cl*(B) ⊆ cl*(A∪B). Therefore, cl*(A)∪cl*(B)⊆cl*(A∪B) (1) cl*(A) and cl*(B) are the closed sets

Therefore, cl*(A)∪cl*(B) is also a closed set. Again, A⊆cl*(A) and B⊆cl*(B)

⇒ A∪B ⊆ cl*(A) ∪ cl*(B)

Thus, cl*(A)∪cl*(B) is a closed set containing A∪B. Since cl*(A∪B) is the smallest closed set containing A∪B.

We have cl*(A∪B) ⊆ cl*(A) ∪ cl*(B) (2)

From (1) and (2),

cl*(A∪B) = cl*(A)∪cl*(B)

Theorem

Union of two τ*- $\ddot{g}$ -closed sets in X is a τ*- $\ddot{g}$ -closed set in X.

Proof

Let A and B be two τ*- $\ddot{g}$ -closed sets.

Let A∪B⊆G, where G is τ*- $\ddot{g}$ -closed.

Since A and B are τ*- $\ddot{g}$ -closed sets, then cl*(A)⊆G and cl*(B)⊆G.

cl*(A)∪cl*(B)⊆G

By the above theorem,

cl*(A)∪cl*(B) = cl*(A∪B)

∴cl*(A∪B)⊆G, where G is τ*-sg-open.

Hence A∪B is a τ*- $\ddot{g}$ -closed set.

Theorem

A subset A of X τ*- $\ddot{g}$ -closed if and only if cl*(A)-A contains no non-empty τ*- $\ddot{g}$ -closed set in X.

Proof

Let A be a τ*- $\ddot{g}$ -closed set.

Suppose that F is a non-empty τ*- $\ddot{g}$ -closed subset of cl*(A) – A.

Now, F⊆cl*(A) – A

Then F⊆cl*(A) ∩ A^C

Since cl*(A) – A = cl*(A)∩A^C

F⊆cl*(A) and F⊆A^C

$\Rightarrow A \subseteq F^C$

Since F^C is a τ*-open set and A is a τ*- $\ddot{g}$ -closed, $cl^*(A) \subseteq F^C$

(i.e) $F \subseteq [cl^*(A)]^C$

Hence $F \subseteq cl^*(A) \cap [cl^*(A)]^C = \phi$

(i.e) $F = \phi$

Which is a contradiction?

Thus cl*(A) – A contains no non-empty τ*-closed set in X.

Conversely, assume that cl*(A)–A contains no non-empty τ*-closed set.

Let $A \subseteq G$, G is a τ*-sg-open.

Suppose that cl*(A) is not contained in G then $cl^*(A) \cap G^C$ is a non-empty, τ*-closed set of cl*(A) – A which is a contradiction.

$\therefore$ $cl^*(A) \subseteq G$, G is τ*-sg-open.

Hence A is τ*- $\ddot{g}$ -closed.

Corollary

A subset A of X is τ*- $\ddot{g}$ -closed if and only if cl*(A) – A contain no non-empty closed set in X.

Proof

The proof follows from the theorem (6.1.15) and the fact that every closed set is τ*- $\ddot{g}$ -closed set in X.

Theorem

If a subset A of X is τ*- $\ddot{g}$ -closed and $A \subseteq B \subseteq cl^*(A)$ then B is τ*- $\ddot{g}$ -closed set in X.

Proof

Let A be a τ*- $\ddot{g}$ -closed set such that $A \subseteq B \subseteq cl^*(A)$.

Let G be a τ*-sg-open set of X such that $B \subseteq G$.

Since A is τ*- $\ddot{g}$ -closed.

We have $cl^*(A) \subseteq G$, whenever $A \subseteq G$ and G is τ*-sg-open

Now, $cl^*(A) \subseteq cl^*(B) \subseteq cl^*[cl^*(A)]$

$= cl^*(A) \subseteq G$

$\therefore$ cl*(B)$\subseteq$G, G is τ*-sg-open set.

Hence B is τ*-$\ddot{g}$-closed set in X.

Theorem

Let A be a τ*-$\ddot{g}$-closed set in (X, τ)

Then A is g-closed if and only if cl*(A) – A is τ*-sg-open.

Proof

Suppose A is g-closed in X. Then cl*(A) = A and so cl*(A) – A = ϕ which is τ*-open in X.

Conversely, suppose cl*(A) – A is τ*-open in X.

Since A is τ*-$\ddot{g}$-closed, by theorem (6.1.15)

cl*(A)–A contains no non-empty τ*-closed set of X. Then cl*(A) – A= ϕ

Hence A is g-closed.

Theorem

For x ∈ X, the set X – {x} is τ*-$\ddot{g}$-closed or τ*-open.

Proof

Suppose X – {x} is not τ*-open. Then X is the only τ*-open set containing X – {x}. This implies cl*(X – {x}) ⊆ X. Hence X – {x} is a τ*-$\ddot{g}$-closed in X.

$\tau^* - \ddot{g} -$ CONTINUOUS AND IN TOPOLOGICAL SPACES

In this section, a new type of functions called $\tau^* - \ddot{g} -$ continuous maps, are introduced and some of its properties are discussed.

Definition

A function f from a topological space X into a topological space (Y, σ) is called $\tau^* - \ddot{g} -$ continuous if $f^{-1}(V)$ is $\tau^* - \ddot{g} -$ closed set in X for every closed set V in Y.

Example

Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {c}} and σ = {Y, ϕ, {b}, {c}, {b, c}, {a, b}}.

Let f: (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = b, f(c) = c. Then f is $\tau^* - \ddot{g} -$ continuous.

Theorem

If a map f: (X, τ) → (Y, σ) from a topological space X into a topological space Y is continuous then it is $\tau^* - \ddot{g} -$ continuous but not conversely.

Proof

Let f: (X, τ) → (Y, σ) be continuous. Let F be any closed set in Y. Then the inverse image $f^{1}(F)$ is a closed set in X. Since every closed set is $\tau^* - \ddot{g} -$ closed, $f^{1}(F)$ is $\tau^* - \ddot{g} -$ closed in X.

∴ F is $\tau^* - \ddot{g} -$ continuous.

Remark

The converse of the above theorem need not be true as seen from the following example.

Example

Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {b}} and σ = {Y, ϕ, {a, b}} Let f: X → Y be an identity map. Hence f is $\tau^* - \ddot{g} -$ continuous. But f is not continuous, since the set {c} is closed in Y but not closed in X.

Theorem

If a map f: (X, τ) → (Y, σ) is g-continuous then it is $\tau^* - \ddot{g} -$ continuous but not conversely.

Proof

Let f: (X, τ) → (Y, σ) be g-continuous Let G be any closed set in Y. Then the inverse image $f^{1}(G)$ is g-closed set in X. Since every g-closed set is $\tau^* - \ddot{g} -$ closed, then $f^{1}(G)$ is $\tau^* - \ddot{g} -$ closed in X.

∴ f is $\tau^* - \ddot{g} -$ continuous.

Remark

The converse need not be true as seen from the following example.

Example

Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {a}} and σ = {Y, ϕ, {b}, {b, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Here f is $\tau^* - \ddot{g} -$ continuous. But f is not g-continuous, since for the closed set {a} in Y, $f^{1}(V)$ = {a} is not g-closed in X.

Let f: (X, τ) → (Y, σ) be a map from a topological space (X, τ) into a topological space (Y, σ) then

- The following statements are equivalent.
- f is $\tau^* - \ddot{g}-$ continuous.
- The inverse image of each open set in Y is $\tau^* - \ddot{g}-$ open in X.
- (ii) If f : $(X, \tau) \to (Y, \sigma)$ is $\tau^* - \ddot{g}-$ continuous, then $f(cl_{\tau^*}(A)) \subseteq cl(f(A))$ for every subset A of X.

Proof

Assume that f: $(X, \tau) \to (Y, \sigma)$ is $\tau^* - \ddot{g}-$ continuous. Let F be open in Y. Then F^C is closed in Y. Since f is $\tau^* - \ddot{g}-$ continuous $f^{1}(F^C)$ is $\tau^* - \ddot{g}-$ closed in X.

But $f^{1}(F^C) = X\text{-}f^{1}(F)$.

Thus. $X\text{-}f^{1}(F)$ is $\tau^* - \ddot{g}-$ closed in X.

$\therefore$ (a) $\Rightarrow$ (b)

Assume that the inverse image of each open set in Y is $\tau^* - \ddot{g}-$ open in X.

Let G be any closed set in Y. Then G^C is open in Y. By assumption, $f^{1}(G^C)$ is $\tau^* - \ddot{g}-$ open in X. But $f^{1}(G^C) = X\text{-}f^{1}(G)$.

$\therefore$ $X\text{-}f^{1}(G)$ is $\tau^* - \ddot{g}-$ open in X and so $f^{1}(G)$ is $\tau^* - \ddot{g}-$ closed in X.

$\therefore$ f is $\tau^* - \ddot{g}-$ continuous.

Hence (b) $\Rightarrow$ (a)

Thus (a) and (b) are equivalent

- Assume that f is $\tau^* - \ddot{g}-$ continuous. Let A be any subset of X, f (A) is a subset of Y. Then cl (f (A)) is a closed subset of Y. Since f is $\tau^* - \ddot{g}-$ continuous, f^{1}(cl (f (A)) is $\tau^* - \ddot{g}-$ closed in x and it containing A. But cl_{τ^*} (A) is the intersection of all $\tau^* - \ddot{g}-$ closed sets containing A.

$\therefore cl_{\tau^*}(A) \subseteq f^{1}(cl(f(A))$

$\Rightarrow f(cl_{\tau^*}(A)) \subseteq cl(f(A))$

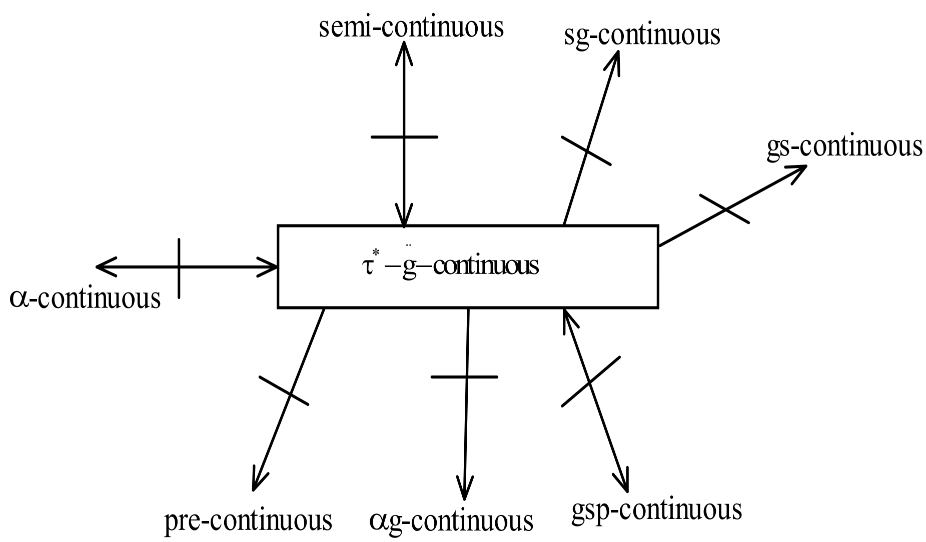
Example

Let X = Y = {a, b, c}. Let f : X → Y be an identity map.

- Let τ = {X, ϕ, {a}, {a, c}} and σ = {Y, ϕ, {a}, {a, b}, {a, c}}. Then f is semi-continuous. But it is not $\tau^{*}-\ddot{g}-$continuous. Since for the closed set V = {c} in Y, $f^{-1}(V)$ = {c} is not $\tau^{*}-\ddot{g}-$closed in X.
- Let τ = {X, ϕ, {c}} and σ = {Y, ϕ, {b}, {c}, {b, c}}. Then f is $\tau^{*}-\ddot{g}-$continuous. But it is not semi-continuous. Since for the closed set V = {a, c} in Y, $f^{-1}(V)$ = {a, c} is not semi-closed in X.
- Let τ = {X, ϕ, {c}} and σ = {Y, ϕ, {b}, {c}, {b, c}}. Then f is $\tau^{*}-\ddot{g}-$continuous. But it is not sg-continuous. Since for the closed set V = {a, c} in Y. $f^{-1}(V)$ = {a, c} is not sg-closed in X.
- Let τ = {X, ϕ, {a}} and σ = {Y, ϕ, {b}, {c}, {b, c}, {a, c}}. Then f is $\tau^{*}-\ddot{g}-$continuous. But it is not gs-continuous. Since for the closed set V = {a} in Y, $f^{-1}(V)$ = {a} is not gs-closed in X.
- Let τ = {X, ϕ, {a, c}} and σ = {Y, ϕ, {a}, {c}, {a, c}, {a, b}}. Then f is gsp-continuous. But it is not $\tau^{*}-\ddot{g}-$continuous. Since for the closed set V = {c} in Y, $f^{-1}(V)$ = {c} is not $\tau^{*}-\ddot{g}-$closed in X.
- Let τ = {X, ϕ, {c}} and σ = {Y, ϕ, {a}, {b}, {a, b}, {b,c}}. The f is $\tau^{*}-\ddot{g}-$continuous. But it is not gsp-continuous. Since for the closed set V = {c} in Y, $f^{-1}(V)$= {c} is not gsp-closed in X.
- Let τ = {X, ϕ, {b}} and σ = {Y, ϕ, {c}, {a, c}}. Then f is $\tau^{*}-\ddot{g}-$continuous. But it is not αg-continuous. Since for the closed set V = {b} in Y, $f^{-1}(V)$ = {b} is not αg-closed in X.
- Let τ = {X, ϕ, {a}} and σ = {Y, ϕ, {c}, {a, c}, {b, c}}. Then f is $\tau^{*}-\ddot{g}-$continuous. But it is not pre-continuous. Since the open set
  V = {b, c} in Y, $f^{-1}(V)$ = {b, c} is not pre-open in X.
- Let τ = {X, ϕ, {c}, {b, c}} and σ = {Y, ϕ, {c}, {a, c}, {b, c}}. Then f is α-continuous but it is not $\tau^{*}-\ddot{g}-$continuous. Since for the closed set
  V = {b} in Y, $f^{-1}(V)$ = {b} is not $\tau^{*}-\ddot{g}-$closed in X.
- Let τ = {X, ϕ, {a}, {b, c}} and σ = {Y, ϕ, {a}, {c}, {a, c}, {b, c}}. Then f is $\tau^{*}-\ddot{g}-$continuous. But it is not α-continuous. Since for the open set V = {a, c} in Y, $f^{-1}(V)$ = {a, c} is not α-open in X.

Remark

From the above discussion, we obtain the following implications.



A $\rightarrow$ B means A implies B, A $\not\rightarrow$ B means A does not imply B, and A $\not\leftrightarrow$ B means A and B are independent.

CONCLUSIONS

The $\tau^* - \ddot{g}-$ closed sets can be used to derive a new homeomorphism, connectedness, compactness and new separation axioms. This concept can be extended to bitopological and fuzzy topological spaces.

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